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Comparison of Explicit and Traditional Algebraic Stress Models of Turbulence

Charles G. Speziale*
Boston University, Boston, Massachusetts 02215

A critical comparison of explicit vs traditional algebraic stress models of turbulence is made in an effort to clear up the confusion that appears to have been generated by the recently published literature on the subject, in which disparate approaches are adopted. It is shown theoretically that the only way that general second-order closures can formally lead to fully explicit algebraic stress models, in a global sense, is in the limit of equilibrium homogeneous turbulence. When these fully explicit models are then applied to turbulent flows that are far from equilibrium, a singularity can arise, which can be removed by a systematic regularization. When solved explicitly either analytically or numerically, the traditional, implicit algebraic stress models are shown to have either multiple solutions or singularities, which tends to explain why they have had problems in applications to complex flows. Thus, it is argued that traditional algebraic stress models are intrinsically ill-behaved and should be abandoned in future applications in favor of regularized, explicit algebraic stress models. It is furthermore argued that these should be based on the homogeneous equilibrium hypothesis, which allows for more general second-order closures to be used to obtain single-valued models.

I. Introduction

RECENTLY, the usefulness of explicit algebraic stress models of turbulence for the calculation of aerodynamic flows of interest has been demonstrated in a variety of test cases.[1,2] These fully explicit algebraic stress models are fairly new,[3] whereas the main ideas behind algebraic stress models are more than 20 years old, including the methodology for their explicit solution by integrity bases methods.[4,5] Consequently, it is not surprising that there appears to exist some confusion in the literature concerning the status and properties of these models. In addition, there seems to be a lack of recognition of the ill-behaved properties of traditional algebraic stress models that are implicit. The purpose of the present paper is to clarify these points. This clarification appears to be currently warranted in light of the recent interesting work of Girimaji[6] (also Durbin, private communication). Because algebraic stress models formally constitute two-equation models, they are computationally less expensive to implement than full second-order closures while building in the physics of these more complex models in the simplified equilibrium limit.

It will be argued in this paper that algebraic stress models can only be made fully explicit—for general second-order closures, in a global sense—in the limit of homogeneous turbulence in equilibrium. This is the approach adopted by Gatski and Speziale,[3] which represented a departure from previous approaches. When the resulting explicit models are then applied to turbulent flows that are far from equilibrium, a singularity can arise. Hence, these models need to be regularized. When the traditional, implicit algebraic stress models are solved explicitly in either an analytical or an iterative numerical way, spurious multiple solutions or a singularity can arise. Consequently, these traditional algebraic stress models are ill-posed and also in need of regularization. Combining this with the fact that algebraic stress models that are fully explicit in a global sense (which is a crucial property for their ultimate application to complex turbulent flows) are only rigorously obtainable from general second-order closures in the homogeneous equilibrium limit, it becomes clear that the new regularized, explicit algebraic stress models are the preferred approach. Although some authors have argued that the need for regularization is a deficiency, it can rather, to the contrary, be a virtue. It allows the model, through a formal Padé approximation that regularizes, to match certain established, far from equilibrium limits without discernibly altering its near equilibrium behavior where such models formally apply.

After first giving a definition of equilibrium turbulence, a summary will be provided on the derivation of explicit algebraic stress models by means of the homogeneous equilibrium hypothesis. Then explicit and traditional algebraic stress models will be compared, and the issues just outlined will be addressed. These points will be discussed in detail in the sections to follow, along with the implications for turbulence modeling.

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*Professor, Aerospace and Mechanical Engineering Department. Member AIAA.

II. Theoretical Background

Incompressible turbulent flows will be considered where the velocity v_i and kinematic pressure P can be decomposed into ensemble mean and fluctuating parts as follows:

$$v_i = \bar{v}_i + u_i, \qquad P = \bar{P} + p \tag{1}$$

The Reynolds stress tensor $\tau_{ij} \equiv \overline{u_i u_j}$ is a solution of the transport equation (cf. Ref. 7)

$$\frac{\mathrm{D}\tau_{ij}}{\mathrm{D}t} = -\tau_{ik}\frac{\partial \bar{v}_j}{\partial x_k} - \tau_{jk}\frac{\partial \bar{v}_i}{\partial x_k} + \Pi_{ij} - \varepsilon_{ij} + \mathcal{D}_{ij}^T + \nu\nabla^2\tau_{ij} \tag{2}$$

where $\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + \bar{\boldsymbol{v}} \cdot \boldsymbol{\nabla}$ is the mean convective derivative and

$$\Pi_{ij} = \overline{p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}, \qquad \varepsilon_{ij} = 2\nu\overline{\left(\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\right)} \tag{3}$$

$$\mathcal{D}_{ij}^T = -\frac{\partial}{\partial x_k}(\overline{u_i u_j u_k} + \overline{p u_i}\delta_{jk} + \overline{p u_j}\delta_{ik}) \tag{4}$$

are, respectively, the pressure–strain correlation, the dissipation rate tensor, and the turbulent transport term.

Homogeneous turbulence achieves an equilibrium state (which is time independent) in the limit as the time $t \to \infty$. This equilibrium is characterized by nondimensional strain rates that are of order one or less, a point that will be discussed in more detail later. Homogeneous turbulence in equilibrium and local regions of inhomogeneous turbulent flows, where there is a production-equals-dissipation equilibrium, satisfy the constraints

$$\frac{\mathrm{D}b_{ij}}{\mathrm{D}t} = 0 \tag{5}$$

$$\mathcal{D}_{ij}^T + \nu\nabla^2\tau_{ij} = 0 \tag{6}$$

where

$$b_{ij} = \frac{\tau_{ij} - \frac{2}{3}K\delta_{ij}}{2K} \tag{7}$$

is the Reynolds stress anisotropy tensor ($K \equiv \frac{1}{2}\overline{u_i u_i}$ is the turbulent kinetic energy). In physical terms, this is an equilibrium for which convective and transport effects can be neglected; it is the basic equilibrium hypothesis used in the derivation of algebraic stress models. However, it is only globally valid for homogeneous turbulent flows that are in equilibrium. Inhomogeneous turbulent flows typically only have very narrow regions where there is a production-equals-dissipation equilibrium, e.g., the logarithmic region in turbulent channel flow where models based on the homogeneous equilibrium hypothesis do well.[3]

It follows directly from Eqs. (5) and (7) that

$$\frac{\mathrm{D}\tau_{ij}}{\mathrm{D}t} = \frac{\tau_{ij}}{K}\frac{\mathrm{D}K}{\mathrm{D}t} \tag{8}$$

and, hence, by making use of the contraction of Eqs. (2) and (6), one can show that

$$\frac{\mathrm{D}\tau_{ij}}{\mathrm{D}t} = (\mathcal{P} - \varepsilon)\frac{\tau_{ij}}{K} \tag{9}$$

where $\mathcal{P} \equiv -\tau_{ij}\partial\bar{v}_i/\partial x_j$ is the turbulence production and $\varepsilon \equiv \frac{1}{2}\varepsilon_{ii}$ is the (scalar) turbulent dissipation rate. The substitution of Eqs. (6) and (9) into Eq. (2) yields the following equilibrium form of the Reynolds stress transport equation:

$$(\mathcal{P} - \varepsilon)\frac{\tau_{ij}}{K} = -\tau_{ik}\frac{\partial\bar{v}_j}{\partial x_k} - \tau_{jk}\frac{\partial\bar{v}_i}{\partial x_k} + \Pi_{ij} - \frac{2}{3}\varepsilon\delta_{ij} \tag{10}$$

where the Kolmogorov assumption of local isotropy[7] given by

$$\varepsilon_{ij} = \tfrac{2}{3}\varepsilon\delta_{ij} \tag{11}$$

has also been applied. By making use of Eq. (7), we can rearrange Eq. (10) into the alternative form in terms of the Reynolds stress anisotropy tensor:

$$(\mathcal{P} - \varepsilon)b_{ij} = -\tfrac{2}{3}K\bar{S}_{ij} - K\left(b_{ik}\bar{S}_{jk} + b_{jk}\bar{S}_{ik} - \tfrac{2}{3}b_{mn}\bar{S}_{mn}\delta_{ij}\right) - K(b_{ik}\bar{\omega}_{jk} + b_{jk}\bar{\omega}_{ik}) + \tfrac{1}{2}\Pi_{ij} \tag{12}$$

where

$$\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial\bar{v}_i}{\partial x_j} + \frac{\partial\bar{v}_j}{\partial x_i}\right), \qquad \bar{\omega}_{ij} = \frac{1}{2}\left(\frac{\partial\bar{v}_i}{\partial x_j} - \frac{\partial\bar{v}_j}{\partial x_i}\right) \tag{13}$$

In virtually all of the commonly used second-order closure models based on Eq. (2), Π_{ij} is modeled in the general form

$$\Pi_{ij} = \varepsilon\mathcal{A}_{ij}(\mathbf{b}) + K\mathcal{M}_{ijkl}(\mathbf{b})\frac{\partial\bar{v}_k}{\partial x_l} \tag{14}$$

(Refs. 8 and 9). The substitution of Eq. (14) into Eq. (12) yields a closed system of algebraic equations for the determination of the Reynolds stress anisotropy in terms of the mean velocity gradients. This constitutes the general form of the traditional algebraic stress models.[4] These traditional algebraic stress models are implicit because the Reynolds stress tensor appears on both sides of the equation. To conduct the algebraic stress model approximation, a linear form for Π_{ij} is typically needed.

It has been shown that the most general nonlinear form of the hierarchy of models (14) for two-dimensional mean turbulent flows in equilibrium systematically reduces to the linear form

$$\Pi_{ij} = -C_1\varepsilon b_{ij} + C_2K\bar{S}_{ij} + C_3K\left(b_{ik}\bar{S}_{jk} + b_{jk}\bar{S}_{ik} - \tfrac{2}{3}b_{mn}\bar{S}_{mn}\delta_{ij}\right) + C_4K(b_{ik}\bar{\omega}_{jk} + b_{jk}\bar{\omega}_{ik}) \tag{15}$$

(Refs. 10 and 11), where only the quadratic return term has been neglected, which is typically small because it is directly proportional to $\varepsilon b_{ik}b_{kj}$, and where C_1–C_4 are constants that depend on the specific model chosen. Speziale et al.[10] showed that a range of strained two-dimensional homogeneous turbulent flows near equilibrium can be collapsed using Eq. (15) with the constants $C_1 = 6.80$, $C_2 = 0.36$, $C_3 = 1.25$, and $C_4 = 0.40$. Out of equilibrium, the first two coefficients are functions of the ratio of production to dissipation, $\mathcal{P}/\varepsilon$, and the second invariant, II, of b_{ij}, which makes the model in Ref. 10 weakly nonlinear. (These coefficients become constants in equilibrium yielding a linear rapid pressure–strain model that is needed for the algebraic stress model approximation.[10]) Hence, by means of the homogeneous equilibrium hypothesis, it becomes possible to systematically conduct the algebraic stress model approximation for nonlinear rapid pressure–strain models.

The direct substitution of Eq. (15) into Eq. (12) yields the equation

$$b_{ij} = \tfrac{1}{2}g\tau\left[\left(C_2 - \tfrac{4}{3}\right)\bar{S}_{ij} + (C_3 - 2)\left(b_{ik}\bar{S}_{jk} + b_{jk}\bar{S}_{ik} - \tfrac{2}{3}b_{mn}\bar{S}_{mn}\delta_{ij}\right) + (C_4 - 2)(b_{ik}\bar{\omega}_{jk} + b_{jk}\bar{\omega}_{ik})\right] \tag{16}$$

where

$$g = [(C_1/2) + (\mathcal{P}/\varepsilon) - 1]^{-1} \tag{17}$$

and $\tau \equiv K/\varepsilon$ is the turbulent time scale. For homogeneous turbulence, the turbulent kinetic energy K and dissipation rate ε are solutions of the transport equations

$$\dot{K} = \mathcal{P} - \varepsilon \tag{18}$$

$$\dot{\varepsilon} = C_{\varepsilon 1}(\varepsilon/K)\mathcal{P} - C_{\varepsilon 2}(\varepsilon^2/K) \tag{19}$$

where $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ are constants that, most recently, have assumed the values of 1.44 and 1.83, respectively (cf. Ref. 10). In Eqs. (18) and (19), a superposed dot represents a time derivative [in homogeneous turbulence, $K = K(t)$ and $\varepsilon = \varepsilon(t)$]. Here, Eq. (18) is exact, whereas Eq. (19) follows from plausible modeling assumptions for homogeneous turbulence near equilibrium.[11] Equations (18) and (19) yield the equilibrium solution

$$\frac{\mathcal{P}}{\varepsilon} = \frac{C_{\varepsilon 2} - 1}{C_{\varepsilon 1} - 1} \tag{20}$$

The homogeneous equilibrium hypothesis, thus, makes use of the standardly adopted dissipation rate transport equation.

If we introduce the dimensionless, rescaled variables

$$S^*_{ij} = \tfrac{1}{2}g\tau(2 - C_3)\bar{S}_{ij}, \qquad \omega^*_{ij} = \tfrac{1}{2}g\tau(2 - C_4)\bar{\omega}_{ij} \tag{21}$$

$$b^*_{ij} = \left[\frac{C_3 - 2}{C_2 - \frac{4}{3}}\right]b_{ij} \tag{22}$$

(see Ref. 3), then Eq. (16) reduces to the simpler form

$$b^*_{ij} = -S^*_{ij} - \left(b^*_{ik}S^*_{jk} + b^*_{jk}S^*_{ik} - \tfrac{2}{3}b^*_{kl}S^*_{kl}\delta_{ij}\right) + b^*_{ik}\omega^*_{kj} + b^*_{jk}\omega^*_{ki} \tag{23}$$

which is linear in b_{ij} by virtue of Eq. (20).

III. Explicit vs Implicit Algebraic Stress Models

The explicit solution to Eq. (23), for two-dimensional mean turbulent flows by integrity bases methods, renders the form[3]

$$b^*_{ij} = -\frac{3}{3 - 2\eta^2 + 6\xi^2} \times \left[S^*_{ij} + S^*_{ik}\omega^*_{kj} + S^*_{jk}\omega^*_{ki} - 2\left(S^*_{ik}S^*_{kj} - \frac{1}{3}S^*_{k\ell}S^*_{k\ell}\delta_{ij}\right)\right] \tag{24}$$

where

$$\eta = \left(S^*_{ij}S^*_{ij}\right)^{\frac{1}{2}}, \qquad \xi = \left(\omega^*_{ij}\omega^*_{ij}\right)^{\frac{1}{2}}$$

For equilibrium turbulent flows, η and ξ are typically less than one, whereas for η, $\xi \gg 1$, we have strongly nonequilibrium turbulent flows that are rapidly distorted (where $\mathcal{P}/\varepsilon$ is also much greater than one). In more familiar terms, Eq. (24) is equivalent to the form

$$\tau_{ij} = \frac{2}{3}K\delta_{ij} - \frac{3}{3 - 2\eta^2 + 6\xi^2}\left[2\alpha_1\frac{K^2}{\varepsilon}\bar{S}_{ij} + \alpha_2\frac{K^3}{\varepsilon^2}(\bar{S}_{ik}\bar{\omega}_{kj} + \bar{S}_{jk}\bar{\omega}_{ki}) - \alpha_3\frac{K^3}{\varepsilon^2}\left(\bar{S}_{ik}\bar{S}_{kj} - \frac{1}{3}\bar{S}_{k\ell}\bar{S}_{k\ell}\delta_{ij}\right)\right] \tag{25}$$

where α_1, α_2, and α_3 are constants related to the coefficients C_1–C_4 and g (see Ref. 3 for more details).

By virtue of the homogeneous equilibrium assumption, which renders the constant value for $\mathcal{P}/\varepsilon$ given in Eq. (20), Eq. (24) is fully explicit. When this expression is applied to inhomogeneous turbulent flows that are out of equilibrium, which is typically done based on a local, homogeneous equilibrium assumption, a singularity can occur through the vanishing of the denominator $(3-2\eta^2+6\xi^2)$ in Eq. (24). Gatski and Speziale[3] introduced the simple regularization

$$\frac{3}{3-2\eta^2+6\xi^2} \approx \frac{3(1+\eta^2)}{3+\eta^2+6\xi^2\eta^2+6\xi^2} \tag{26}$$

which is obtained by a Taylor expansion that is a systematic approximation for η and ξ sufficiently less than one. For turbulent flows that are near equilibrium, where Eq. (24) is formally valid and $\eta,\ \xi<1$, this approximation yields results that are indistinguishable from the original expression $3/(3-2\eta^2+6\xi^2)$, all within the framework of a model that is regular and computable for all values of η and ξ. More recently, Speziale and Xu[12] conducted a formal Padé approximation of the coefficient $3/(3-2\eta^2+6\xi^2)$ that regularizes while building in some limited agreement with the rapid distortion theory solution for homogeneous shear flow that is strongly sheared where $\eta,\ \xi \gg 1$. Abid et al.[2] conducted a heuristic Padé approximation that takes the alternative, pragmatic view of maintaining a finite lower bound for the eddy viscosity for numerical robustness. [On the other hand, Eq. (26) yields a vanishing eddy viscosity as $\eta,\ \xi \to \infty$, which can be numerically destabilizing.] Equation (24), with a regularized expression for the strain-dependent coefficient $3/(3-2\eta^2+6\xi^2)$, constitutes the general form of the regularized, explicit algebraic stress models based on the homogeneous equilibrium hypothesis. They are in the form of anisotropic eddy viscosity models with strain-dependent coefficients.

Traditional algebraic stress models have been based on the implicit equation (12) wherein the Launder et al.[13] model is typically used [in Eq. (15), this corresponds to $C_1=3.0$, $C_2=0.8$, $C_3=1.75$, and $C_4=1.31$]. Since $\mathcal{P}\equiv -2Kb_{ij}\bar{S}_{ij}$, it follows that Eq. (12) actually constitutes a coupled system of quadratic equations for the determination of b_{ij} when the homogeneous equilibrium hypothesis, with its constant value for $\mathcal{P}/\varepsilon$, is not invoked. This leads to the so-called consistency condition for $\mathcal{P}/\varepsilon$. Multiplying Eq. (24) by the quantity $-2[(C_2-\frac{4}{3})/(C_3-2)]\tau\bar{S}_{ij}$ yields the following equation for $\mathcal{P}/\varepsilon$:

$$\frac{\mathcal{P}}{\varepsilon} = -\frac{[3(C_2-\frac{4}{3})]/[(C_3-2)^2 g]}{3-2\eta^2+6\xi^2}\eta^2 \tag{27}$$

where we have used the fact that $S^*_{ij}S^*_{jk}S^*_{ki}=0$ for two-dimensional mean turbulent flows. Because, in general,

$$\eta,\ \xi \propto \frac{1}{\frac{1}{2}C_1+\mathcal{P}/\varepsilon-1}$$

as a result of Eqs. (17) and (21), it follows that Eq. (27) leads to a cubic equation for $\mathcal{P}/\varepsilon$ (Ref. 6). This cubic equation has three multiple roots (two of which are usually physically spurious), which tends to explain why the traditional algebraic stress models have had problems in applications to complex turbulent flows.

Equation (12) can be linearized in b_{ij} by solving it iteratively as follows:

$$\begin{aligned}
[(\mathcal{P}/\varepsilon)^{(n)}-1]b_{ij}^{(n+1)} &= -\frac{2}{3}\tau^{(n)}\bar{S}_{ij}^{(n+1)} \\
&- \tau^{(n)}\left[b_{ik}^{(n+1)}\bar{S}_{jk}^{(n+1)} + b_{jk}^{(n+1)}\bar{S}_{ik}^{(n+1)} - \frac{2}{3}b_{mn}^{(n+1)}\bar{S}_{mn}^{(n+1)}\delta_{ij}\right] \\
&- \tau^{(n)}\left[b_{ik}^{(n+1)}\bar{\omega}_{jk}^{(n+1)} + b_{jk}^{(n+1)}\bar{\omega}_{ik}^{(n+1)}\right] + \frac{1}{2}\frac{\Pi_{ij}^{(n+1)}}{\varepsilon^{(n)}}
\end{aligned} \tag{28}$$

as has been done in many computational applications of algebraic stress models (cf. Ref. 14). In Eq. (28), K and ε in $\Pi_{ij}^{(n+1)}$ are evaluated at the preceding time step (n). This equation, which is usually solved numerically by a matrix inversion, has the exact solution

$$\begin{aligned}
b_{ij}^{*(n+1)} = -\frac{3}{3-2\eta^2+6\xi^2}\Bigg\{ & S_{ij}^{*(n+1)} + S_{ik}^{*(n+1)}\omega_{kj}^{*(n+1)} + S_{jk}^{*(n+1)} \\
& \times \omega_{ki}^{*(n+1)} - 2\left[S_{ik}^{*(n+1)}S_{kj}^{*(n+1)} - \frac{1}{3}S_{k\ell}^{*(n+1)}S_{k\ell}^{*(n+1)}\delta_{ij}\right]\Bigg\}
\end{aligned} \tag{29}$$

where

$$S_{ij}^{*(n+1)} = \tfrac{1}{2}g^{(n)}\tau^{(n)}(2-C_3)\bar{S}_{ij}^{(n+1)}$$

$$\omega_{ij}^{*(n+1)} = \tfrac{1}{2}g^{(n)}\tau^{(n)}(2-C_4)\bar{\omega}_{ij}^{(n+1)}$$

$$b_{ij}^{*(n+1)} = \left(\frac{C_3-2}{C_2-\frac{4}{3}}\right)b_{ij}^{(n+1)}$$

given that $\eta=[S_{ij}^{*(n+1)}S_{ij}^{*(n+1)}]^{1/2}$ and $\xi=[\omega_{ij}^{*(n+1)}\omega_{ij}^{*(n+1)}]^{1/2}$, whereas $g^{(n)}$ is evaluated with the value of $(\mathcal{P}/\varepsilon)^{(n)}$ at the preceding iteration to linearize the system in b_{ij}. It is thus clear that the system can become singular when solved numerically in this fashion through the vanishing of the denominator $(3-2\eta^2+6\xi^2)$ in the solution (29). The traditional implicit algebraic stress models are, thus, intrinsically ill-posed when solved iteratively.

Recently, Girimaji[6] applied the consistency condition for $\mathcal{P}/\varepsilon$ and solved the cubic equation (27). (Although it was not directly provided, this cubic equation was apparently solved numerically by Pope.[5]) This equation has a general solution of the form

$$\frac{\mathcal{P}}{\varepsilon} = \frac{\mathcal{P}}{\varepsilon}(\bar{S}_{ij},\ \bar{\omega}_{ij}) \tag{30}$$

which has three distinct roots due to the fact that Eq. (27) is a cubic equation. Girimaji[6] regularized the system by maintaining one root while discarding the remaining two, which were deemed unphysical. This solution can then be substituted into Eq. (12), rendering a linear system of equations for b_{ij} that has a unique solution, which is regular for all values of η and ξ. Unfortunately, however, the resulting solution for b_{ij} (Ref. 6) is rather complicated and cumbersome to use in applications. Furthermore, one can ask the question as to what benefit is gained from this approach. It is only for homogeneous turbulent flows in equilibrium that the pivotal rapid pressure–strain correlation for general second-order closures systematically collapses to the general linear form given in Eq. (15), which is usually required to conduct the algebraic stress model approximation. Thus, one is formally restricted by this approach to linear models like that of Launder et al.,[13] which has been shown to perform in an inferior manner to the newer nonlinear models, such as the model in Ref. 10, in a variety of benchmark flows. Additionally, it is rather questionable to apply a consistency condition for $\mathcal{P}/\varepsilon$ out of equilibrium where the algebraic stress model approximation does not formally apply in the first place. When one combines these points with the fact that it is not completely clear which are the physically spurious roots in complex flows, the author sees no advantage to this approach.

IV. Concluding Remarks

Explicit and traditional algebraic stress models of turbulence have been critically compared in an effort to clarify the basic properties and structure of these models. It was definitively demonstrated that the traditional algebraic stress models, which are implicit in nature, are fundamentally ill-behaved. They can give rise to either multiple solutions or a singular system of equations in applications. Hence, two major conclusions have been arrived at in this paper.

1) There appears to be no question that the traditional algebraic stress models, which are ill-posed, should be abandoned in future applications in favor of regularized, explicit algebraic stress models.

2) In the opinion of the author, this should be done with models based on the homogeneous equilibrium hypothesis discussed herein, which allows general second-order closures that are nonlinear in the rapid pressure–strain correlation to be systematically used to extend the range of validity of the models.

Models for the pivotal pressure–strain correlation have been developed based on the underlying assumption that the turbulence is homogeneous and in equilibrium.[11] Furthermore, the homogeneous equilibrium hypothesis constitutes the only assumption by which the algebraic stress model approximation is globally valid. Although these models are derived for homogeneous turbulence, they are then bodily extended to inhomogeneous flows. The alternative of applying the consistency condition for $\mathcal{P}/\varepsilon$, with its cubic equation, is fraught with problems as discussed herein. (A Renormalization Group-based subgrid scale model was developed that ultimately had to be substantially modified because the eddy viscosity was the solution of a cubic equation.[15]) The regularization that the models must be subjected to, which can be systematically accomplished via a Padé approximation, rather than being a deficiency, is a virtue that allows the model to match certain far from equilibrium limits where it would not normally apply, thus extending its range of applicability. The traditional algebraic stress models are only formally valid for turbulent flows that are near equilibrium and can yield erroneous predictions when the departures from equilibrium are large. Thus, the kind of extensions discussed in this paper are needed. Recently published aerodynamic applications[1,2] based on this new approach are quite promising, and further tests are currently underway.

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References

[1] Abid, R., Rumsey, C., and Gatski, T. B., "Prediction of Non-Equilibrium Turbulent Flows with Explicit Algebraic Stress Models," *AIAA Journal*, Vol. 33, No. 11, 1995, pp. 2026–2031.

[2] Abid, R., Morrison, J. H., Gatski, T. B., and Speziale, C. G., "Prediction of Aerodynamic Flows with a New Explicit Algebraic Stress Model," *AIAA Journal*, Vol. 34, No. 12, 1996, pp. 2632–2635.

[3] Gatski, T. B., and Speziale, C. G., "On Explicit Algebraic Stress Models for Complex Turbulent Flows," *Journal of Fluid Mechanics*, Vol. 254, 1993, pp. 59–78.

[4] Rodi, W., "A New Algebraic Relation for Calculating the Reynolds Stresses," *ZAMM*, Vol. 56, 1976, pp. T219–T221.

[5] Pope, S. B., "A More General Effective Viscosity Hypothesis," *Journal of Fluid Mechanics*, Vol. 72, 1975, pp. 331–340.

[6] Girimaji, S., "Fully-Explicit and Self-Consistent Algebraic Reynolds Stress Model," *Theoretical and Computational Fluid Dynamics*, Vol. 8, 1996, pp. 387–402.

[7] Hinze, J. O., *Turbulence*, 2nd ed., McGraw–Hill, New York, 1975.

[8] Reynolds, W. C., "Fundamentals of Turbulence for Turbulence Modeling and Simulation," *Lecture Notes for Von Kármán Institute*, AGARD Lecture Series 86, 1987, pp. 1–66.

[9] Speziale, C. G., "Analytical Methods for the Development of Reynolds-Stress Closures in Turbulence," *Annual Review of Fluid Mechanics*, Vol. 23, 1991, pp. 107–157.

[10] Speziale, C. G., Sarkar, S., and Gatski, T. B., "Modeling the Pressure-Strain Correlation of Turbulence: An Invariant Dynamical Systems Approach," *Journal of Fluid Mechanics*, Vol. 227, 1991, pp. 245–272.

[11] Speziale, C. G., "Modeling of Turbulent Transport Equations," *Simulation and Modeling of Turbulent Flows*, edited by T. B. Gatski, M. Y. Hussaini, and J. L. Lumley, Oxford Univ. Press, Oxford, England, UK, 1996, pp. 185–242.

[12] Speziale, C. G., and Xu, X. H., "Towards the Development of Second-Order Closure Models for Non-Equilibrium Turbulent Flows," *International Journal of Heat and Fluid Flow*, Vol. 17, No. 3, 1996, pp. 238–244.

[13] Launder, B. E., Reece, G. J., and Rodi, W., "Progress in the Development of a Reynolds Stress Turbulence Closure," *Journal of Fluid Mechanics*, Vol. 68, 1975, pp. 537–566.

[14] Rodi, W., *Turbulence Models and Their Application in Hydraulics.*, International Association of Hydraulics Research, Delft, The Netherlands, 1980.

[15] Yakhot, V., and Orszag, S. A., "Renormalization Group Analysis of Turbulence I. Basic Theory," *Journal of Scientific Computing*, Vol. 1, No. 1, 1986, pp. 3–51.

F. W. Chambers
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